

(2+1)-Dimensional Gravity in Weyl Integrable Spacetime

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We investigate (2+1)-dimensional gravity in a Weyl integrable spacetime (WIST). We show that, unlike general relativity, this scalar-tensor theory has a Newtonian limit for any dimension $n \geq 3$ and that in three dimensions the congruence of world lines of particles of a pressureless fluid has a non-vanishing geodesic deviation. We present and discuss a class of static vacuum solutions generated by a circularly symmetric matter distribution that for certain values of the parameter ω corresponds to a space-time with a naked singularity at the center of the matter distribution. We interpret all these results as being a direct consequence of the space-time geometry.

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I. INTRODUCTION

During the past three decades a great deal of effort has gone to the investigation into gravity theories in (2+1) space-time dimensions [1]. It seems that part of this interest has been motivated by the fact that in this dimensionality Einstein gravity presents some odd peculiarities. First, it has no propagating gravity modes, which implies that its quantum version contains no gravitons. This is due to the fact that in (2+1) dimensions space-time is flat outside sources. For point sources gravity manifests itself as global topological defects rather than local geometrical curvature. Secondly, in three-dimensional space-time Einstein theory does not reduce to Newtonian gravity in the static weak-field regime [2]. Another interest in (2+1)- dimensional gravity comes from the fact that studying classical models in lower dimensions has often been helpful to understand their quantum version [3].

The failure of (2+1)-dimensional Einstein gravity to provide a relativistic generalization of two-dimensional Newtonian gravity has led some authors to investigate the same problem in other theories of gravity. It has been proved that at least in two distinct gravitational theories the Newtonian limit may be recovered. These are the Brans-Dicke theory and the teleparallel gravity, a metric theory in which gravity is purely due to torsion [4]. Another approach to address this question is to consider (2+1)- dimensional gravity as obtained through a dimensional reduction of four-dimensional Einstein gravity [5]. The basic motivation in many of these attempts is to seek alternative ways beyond the scope of general relativity to gain more insight into some problems that do not seem to have a satisfactory solution in the context of Einstein theory. With the same idea in mind we consider the subject in the context of another theory of gravity, namely, the Weyl integrable space-time theory (WIST) [6]. In this approach, the geometry of space-time is not Riemannian, but corresponds to what is called a Weyl integrable geometry, a particular version of the geometry developed by H. Weyl in 1918 in connection with his gravitational theory [7]. We show that, in addition to leading to a Newtonian limit, WIST in (2+1) dimensions presents some interesting properties that are not shared by Einstein theory, such as geodesic deviation between particles in a dust distribution and the existence of naked singularities.

The gravitational theory developed by Weyl arose as one of the first attempts to unify gravity and electromagnetism. Its geometrical structure is based on one of the simplest generalizations of Riemannian geometry. As is well known, the kind of geometrical structure conceived by H. Weyl, although admirably elegant and ingenious, turned out to be unsuitable as a physical theory. It came from Einstein the first objections to the theory, who argued that in a non-integrable Weyl geometry the existence of sharp spectral lines in the presence of an electromagnetic field would not be possible since atomic clocks would depend on their past history [8]. Later, it was found that a variant of Weyl geometries, known as Weyl integrable geometry, does not suffer from the drawback pointed out by Einstein, and since then has attracted the attention of some cosmologists [6]. In the opinion of some authors, Weyl theory "contains a

suggestive formalism and may still have the germs of a future fruitful theory ” [9].

The paper is organized as follows. In Sections 2 and 3 we give a brief account of the Weyl geometry and introduce the formalism of a specific theory of gravity known as Weyl integrable space-time (WIST). The (2+1)-dimensional version of this theory is presented in Section 4. In Section 5, we show that WIST has a Newtonian limit for any dimension $n \geq 3$. We proceed to Section 6 to prove that, unlike (2+1)-dimensional general relativity, in three-dimensional Weyl gravity the congruence of world lines of particles of a pressureless fluid has a non-vanishing geodesic deviation. In Section 7, we present and discuss a class of static vacuum solutions generated by a circularly symmetric matter distribution. Finally, we conclude with some remarks in Section 8.

II. WEYL GEOMETRY

We can easily define Weyl geometry by its essential difference from the Riemann geometry: while in the latter one makes the assumption that the covariant derivative of the metric tensor g is zero, in Weyl geometry we have

$$\nabla_\alpha g_{\beta\lambda} = \sigma_\alpha g_{\beta\lambda} \quad (1)$$

where σ_α denotes the components of a one-form field σ with respect to a local coordinate basis¹. This represents a generalization of the Riemannian condition of compatibility between the connection ∇ and g , which is equivalent to require the length of a vector to remain unaltered by parallel transport [8]. If $\sigma = d\phi$, where ϕ is a scalar field, then we have what is called an integrable Weyl geometry. A differentiable manifold M endowed with a metric g and a Weyl scalar field ϕ is usually referred to as a *Weyl frame*. It is interesting to note that the Weyl condition (1) remains unchanged when we go to another Weyl frame $(M, \bar{g}, \bar{\phi})$ by performing the following simultaneous transformations in g and ϕ :

$$\bar{g} = e^f g, \quad (2)$$

$$\bar{\phi} = \phi + f, \quad (3)$$

where f is a scalar function defined on M .

Quite analogously to Riemann geometry, the condition (1) is sufficient to determine the Weyl connection ∇ in terms of the metric g and the Weyl one-form field σ . Indeed, a straightforward calculation shows that one can express the components of the affine connection with respect to an arbitrary vector basis completely in terms of the components of g and σ :

$$\Gamma_{\beta\lambda}^\alpha = \{\alpha_{\beta\lambda}\} - \frac{1}{2} g^{\alpha\mu} [g_{\mu\beta} \sigma_\lambda + g_{\mu\lambda} \sigma_\beta - g_{\beta\lambda} \sigma_\mu], \quad (4)$$

where $\{\alpha_{\beta\lambda}\}$ represents the Christoffel symbols.

A clear geometrical insight into the properties of Weyl parallel transport is given by the following proposition: Let M be a differentiable manifold with an affine connection ∇ , a metric g and a Weyl field of one-forms σ . If ∇ is compatible with g in the Weyl sense, i.e. if (1) holds, then for any smooth curve $\alpha = \alpha(\lambda)$ and any pair of two parallel transported vector fields V and U along α , we have

$$\frac{d}{d\lambda} g(V, U) = \sigma\left(\frac{d}{d\lambda}\right) g(V, U) \quad (5)$$

where $\frac{d}{d\lambda}$ denotes the vector tangent to α .

If we integrate the above equation along the curve α , starting from a point $P_0 = \alpha(\lambda_0)$, then we obtain

$$g(V(\lambda), U(\lambda)) = g(V(\lambda_0), U(\lambda_0)) e^{\int_{\lambda_0}^{\lambda} \sigma(\frac{d}{d\rho}) d\rho} \quad (6)$$

¹ Throughout this paper our convention is that Greek indices take values from 0 to $n-1$, n being the space-time dimension.

Putting $U = V$ and denoting by $L(\lambda)$ the length of the vector $V(\lambda)$ at an arbitrary point $P = \alpha(\lambda)$ of the curve, then it is easy to see that in a local coordinate system $\{x^\alpha\}$ the equation (5) reduces to

$$\frac{dL}{d\lambda} = \frac{\sigma_\alpha}{2} \frac{dx^\alpha}{d\lambda} L$$

Consider the set of all closed curves $\alpha : [a, b] \in R \rightarrow M$, i.e, with $\alpha(a) = \alpha(b)$. Then, we have the equation

$$g(V(b), U(b)) = g(V(a), U(a)) e^{\int_a^b \sigma(\frac{d}{d\lambda}) d\lambda}.$$

Now, it is the integral $\int_a^b \sigma(\frac{d}{d\lambda}) d\lambda$ that is responsible for the difference between the readings of two identical atomic clocks following different paths. It follows from Stokes' theorem that if σ is an exact form, that is, if there exists a scalar function ϕ , such that $\sigma = d\phi$, then

$$\oint \sigma(\frac{d}{d\lambda}) d\lambda = 0$$

for any loop. In other words, in this case the integral $e^{\int_{\lambda_0}^\lambda \sigma(\frac{d}{d\lambda}) d\lambda}$ does not depend on the path. Since it is this integral that regulates the way atomic clocks run this variant of Weyl geometry does not suffer from the flaw pointed out by Einstein, and we have what is often called in the literature a *Weyl integrable manifold*. In the next section we shall consider a theory of gravity (WIST) formulated in a Weyl integrable manifold.

III. WEYL INTEGRABLE SPACE-TIME THEORY IN N DIMENSIONS

In Weyl integrable space-time theory it is assumed that, in n dimensions, the dynamics of the gravitational field is given by the following action [6]:

$$^{(n)}\mathcal{S} = \int d^n x \sqrt{|g|} \left[\mathcal{R} + \omega \phi_{,\alpha} \phi^{,\alpha} + \kappa_n e^{-n\phi/2} L_m \right] \quad (7)$$

where ω is an arbitrary coupling constant, $\phi_{,\alpha}$ denotes the derivative $\frac{\partial \phi}{\partial x^\alpha}$ of the Weyl scalar field ϕ , \mathcal{R} is the Weylian Ricci scalar, L_m is the Lagrangian of matter, κ_n denotes the Einstein constant in n dimensions and, as usual, $|g|$ indicates the absolute value of the determinant of the metric tensor². It is important to note here that L_m is constructed by following the so-called Weyl minimal coupling prescription [10]. This means that it will be assumed that L_m depends on ϕ , $g_{\mu\nu}$ and the matter fields, here generically designated by ξ , its form being obtained from the special theory of relativity through the "minimum coupling" prescription $\eta_{\mu\nu} \rightarrow e^{-\phi} g_{\mu\nu}$ and $\partial_\mu \rightarrow \nabla_\mu$, where ∇_μ denotes the covariant derivative with respect to the Weyl affine connection. If we designate the Lagrangian of the matter fields in special relativity by $L_m^{sr} = L_m^{sr}(\eta, \xi, \partial\xi)$, then the form of L_m will be given by the rule $L_m(g, \phi, \xi, \nabla\xi) \equiv L_m^{sr}(e^{-\phi} g, \xi, \nabla\xi)$. As it can be easily seen, these rules also ensure the invariance under Weyl transformations of part of the action that is responsible for the coupling of matter with the gravitational field, and, at the same time, reproduce the principle of minimal coupling adopted in general relativity when we set $\phi = 0$, that is, when we go to the Riemann frame by a Weyl transformation.

We can also express the above action as

$$^{(n)}\mathcal{S} = \int d^n x \sqrt{|g|} \left[R + \frac{(n-1)(n-2) + 4\omega}{4} \phi_{,\alpha} \phi^{,\alpha} + \kappa_n e^{-n\phi/2} L_m \right],$$

where R represents the scalar curvature evaluated with respect to the Riemannian connection³. Let us now recall how the energy-momentum tensor $T_{\mu\nu}(\phi, g, \xi, \nabla\xi)$ is defined in WIST gravity. In an arbitrary Weyl frame $T_{\mu\nu}(\phi, g, \xi, \nabla\xi)$ is defined by the formula

$$\delta \int d^n x \sqrt{|g|} e^{-n\phi/2} L_m(g_{\mu\nu}, \phi, \xi, \nabla\xi) = \int d^n x \sqrt{|g|} e^{-n\phi/2} T_{\mu\nu}(\phi, g_{\mu\nu}, \xi, \nabla\xi) \delta(e^\phi g^{\mu\nu}), \quad (8)$$

² Throughout this paper we shall adopt the following convention in the definition of the Riemann and Ricci tensors: $R^\alpha_{\mu\beta\nu} = \Gamma^\alpha_{\beta\mu,\nu} - \Gamma^\alpha_{\mu\nu,\beta} + \Gamma^\alpha_{\rho\nu} \Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\rho\beta} \Gamma^\rho_{\nu\mu}$; $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$. In this convention, we shall write the Einstein equations as $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa T_{\mu\nu}$, with $\kappa = \frac{8\pi G}{c^4}$.

³ In order to keep the kinetic term in the Lagrangian we shall assume throughout the paper that $\omega \neq -\frac{(n-1)(n-2)}{4}$.

where the variation on the left-hand side must be carried out simultaneously with respect to both $g_{\mu\nu}$ and ϕ . In order to see that the above definition makes sense, it must be understood that the left-hand side of the equation (8) can always be put in the same form of the right-hand side of the same equation. This can easily be seen from the fact that $\delta L_m = \frac{\partial L_m}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial L_m}{\partial \phi} \delta \phi = \frac{\partial L_m}{\partial (e^\phi g^{\mu\nu})} \delta (e^\phi g^{\mu\nu})$ and that $\delta(\sqrt{|g|} e^{-n\phi/2}) = -\frac{1}{2} \sqrt{|g|} e^{-\phi(1+n/2)} g_{\mu\nu} \delta(e^\phi g^{\mu\nu})$.

Varying the action $^{(n)}S$ with respect to the metric $g_{\alpha\beta}$ and to the Weyl scalar field ϕ , we obtain, respectively, the following equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{(n-1)(n-2) + 4\omega}{4} \left[\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} \right] = -\kappa_n T_{\mu\nu} e^{(1-n/2)\phi}, \quad (9)$$

$$\square \phi = \frac{2\kappa_n}{(n-1)(n-2) + 4\omega} e^{(1-n/2)\phi} T \quad (10)$$

where the symbol $^{(n)}\square$ denotes the n -dimensional D'Alembertian operator with respect to the Riemannian connection, and $T = g^{\mu\nu} T_{\mu\nu}$.

IV. WEYL INTEGRABLE THEORY OF GRAVITY IN (2+1)- DIMENSIONAL SPACE-TIME

Let us now consider WIST in a (2+1)-dimensional space-time in the absence of matter. In this case, if $n = 3$ (9) and (10) become

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{(1+2\omega)}{2} \left[\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha} \right] = 0, \quad (11)$$

$$\square \phi = 0, \quad (12)$$

recalling that $\omega \neq -\frac{1}{2}$. On the other hand, if we take the trace of the equation (9) we shall get

$$R = -\frac{1}{2} (2\omega + 1) \phi_{,\alpha} \phi^{,\alpha}. \quad (13)$$

Substituting (13) into equation (11) leads to

$$R_{\mu\nu} = -\frac{1}{2} (2\omega + 1) \phi_{,\mu} \phi_{,\nu}. \quad (14)$$

Let us now recall the well-known mathematical fact which says that for $n > 3$ the Riemann tensor $R_{\lambda\mu\nu\kappa}$ can be decomposed in terms of the Ricci tensor $R_{\mu\nu}$ the scalar curvature R and the Weyl tensor $W_{\lambda\mu\nu\kappa}$. However, if $n = 3$, then, because $W_{\lambda\mu\nu\kappa}$ vanishes identically, we have the following expression for $R_{\lambda\mu\nu\kappa}$ [12]:

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} + g_{\mu\kappa} R_{\lambda\nu} - g_{\mu\nu} R_{\lambda\kappa} - \frac{R}{2} (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}). \quad (15)$$

By taking into account (13) and (14) we can express (15) as

$$R_{\lambda\mu\nu\kappa} = k_n e^{-\phi/2} [g_{\lambda\kappa} T_{\mu\nu} + g_{\mu\nu} T_{\lambda\kappa} - g_{\lambda\nu} T_{\mu\kappa} - g_{\mu\kappa} T_{\lambda\nu} + (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}) T] + \Psi_{\lambda\mu\nu\kappa} \quad (16)$$

where

$$\Psi_{\lambda\mu\nu\kappa} = -\frac{1}{2} (2\omega + 1) [g_{\lambda\nu} \phi_{,\mu} \phi_{,\kappa} + g_{\mu\kappa} \phi_{,\lambda} \phi_{,\nu} - g_{\lambda\kappa} \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \phi_{,\lambda} \phi_{,\kappa}] + \frac{1}{4} (2\omega + 1) (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}) \phi_{,\alpha} \phi^{,\alpha} \quad (17)$$

The equation (16) means that even in the absence of matter, except for $\omega = -\frac{1}{2}$, the space-time is not necessarily Riemann flat as it depends on the Weyl scalar field. Thus, unlike general relativity in (2+1) dimensions, in WIST gravity the gravitational field does not necessarily vanish outside the sources. Likewise, the curvature tensor $\mathcal{R}^{\alpha}_{\mu\nu\kappa}$ calculated with the Weyl connection $\Gamma^{\alpha}_{\beta\lambda}$ does not vanish in the absence of matter as it is given by

$$\mathcal{R}^{\alpha}_{\mu\nu\kappa} = g^{\alpha\lambda} R_{\lambda\mu\nu\kappa} - e^{-\phi} g_{\mu\lambda} \delta^{[\alpha}_{[\nu} Q^{\lambda]}_{\kappa]}, \quad (18)$$

with $Q^{\alpha}_{\beta} = 4e^{\phi/2} (e^{\phi/2})_{,\beta;\lambda} g^{\alpha\lambda} - 2 (e^{\phi/2})_{,\mu} (e^{\phi/2})_{,\nu} g^{\mu\nu} \delta^{\alpha}_{\beta}$, and $\delta^{[\alpha}_{[\nu} Q^{\lambda]}_{\kappa]}$ denotes antisymmetrization with respect to both upper and lower indices. In the next section, we shall investigate the Newtonian limit of Weyl gravity in the weak-field regime.

V. THE NEWTONIAN LIMIT

A metric theory of gravity is said to possess a Newtonian limit in the non-relativistic weak-field regime if one can derive Newton's second law from the geodesic equations as well as Poisson's equation from the gravitational field equations. Let us now proceed to examine whether Weyl gravity fulfills these requirements. The method we shall employ here to treat this problem is standard and can be found in most textbooks on general relativity (see, for instance, ref. [9]).

Since in Newtonian mechanics the space geometry is Euclidean, a weak gravitational field in a geometric theory of gravity should manifest itself as a metric phenomenon through a slight perturbation of the Minkowskian space-time metric. Thus we consider a time-independent metric tensor of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad (19)$$

where $\eta_{\mu\nu}$ denotes Minkowski metric tensor, ϵ is a small parameter and the term $\epsilon h_{\mu\nu}$ represents a very small time-independent perturbation due to the presence of some matter configuration. Since we are working in the non-relativistic regime we shall suppose that the velocity V of a particle moving along a geodesic is much less than c , so that the parameter $\beta = \frac{V}{c}$ will be regarded as very small; hence in our calculations only first-order terms in ϵ and β will be kept. The same kind of approximation will be assumed to hold with respect to the Weyl scalar field ϕ , which will be supposed to be static and very small, i.e., of the same order as ϵ , and to emphasise this fact we shall write $\phi = \epsilon\varphi$, where φ is finite.

If we adopt the Galilean coordinates of special relativity we can write the line element defined by (19) as

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - \epsilon h_{\mu\nu} dx^\mu dx^\nu,$$

which leads, in our approximation, to

$$\left(\frac{ds}{dt}\right)^2 \cong c^2(1 + \epsilon h_{00}) \quad (20)$$

We now apply the same approximation to the geodesic equations

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \quad (21)$$

recalling that the symbol $\Gamma_{\alpha\beta}^\mu$ represents the components of the Weyl affine connection. From (4) it is easy to see that, to first order in ϵ , we have

$$\Gamma_{\mu\nu}^\alpha = \frac{\epsilon}{2} n^{\alpha\lambda} [h_{\lambda\mu,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda} + n_{\mu\nu}\varphi_{,\lambda} - n_{\lambda\mu}\varphi_{,\nu} - n_{\lambda\nu}\varphi_{,\mu}] \quad (22)$$

It is not difficult to see that, unless $\mu = \nu = 0$, the product $\Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}$ is of order $\epsilon\beta$ or higher. In this way, the geodesic equations (21) become, to first order in ϵ and β

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{00}^\mu \left(\frac{dx^0}{ds}\right)^2 = 0$$

By taking into account (20) the above equation may be written as

$$\frac{d^2 x^\mu}{dt^2} + c^2 \Gamma_{00}^\mu = 0 \quad (23)$$

Clearly for $\mu = 0$ the equation (23) reduces to an identity. On the other hand, if μ is a spatial index, a simple calculation gives us $\Gamma_{00}^i = -\frac{\epsilon}{2} \eta^{ij} \frac{\partial}{\partial x^j} (h_{00} - \varphi)$, hence the geodesic equation in this approximation becomes, in three-dimensional vector notation

$$\frac{d^2 \vec{X}}{dt^2} = -\frac{\epsilon}{2} c^2 \vec{\nabla} (h_{00} - \varphi),$$

which is simply Newton's equation of motion in a classical gravitational field provided we identify the scalar gravitational potential as

$$U = \frac{\epsilon c^2}{2} (h_{00} - \varphi). \quad (24)$$

It is interesting to note here the presence of the Weyl field φ in the equation above. It is the combination $h_{00} - \varphi$ that make up the Newtonian potential.

Let us now turn our attention to the Newtonian limit of the field equations. As we have seen previously, in n -dimensions the field equations of Weyl gravity in the presence of matter are given by (9) and (10). It is now convenient to recast the equation (9) into the form

$$R_{\mu\nu} = -\kappa_n e^{(1-n/2)\phi} (T_{\mu\nu} - g_{\mu\nu} \frac{T}{n-2}) - \frac{(n-1)(n-2) + 4\omega}{4} \phi_{,\mu} \phi_{,\nu} \quad (25)$$

In the weak-field approximation, i.e. when $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$, it is easy to show that to first order in ϵ , we have $R_{00} = -\frac{1}{2}\nabla^2 \epsilon h_{00}$, where ∇^2 denotes the Laplacian operator in n -dimensional flat space-time. On the other hand, because we are assuming a static regime $\phi_{,0} = 0$, so the equation (25) for $\mu = \nu = 0$ now reads

$$\nabla^2 h_{00} = -\kappa_n \left(T_{00} - \frac{g_{00}T}{n-2} \right) e^{(1-n/2)\phi}$$

For a perfect fluid configuration, according to our minimal coupling prescription, we have $T_{\mu\nu} = e^{-2\phi}[(\rho c^2 + p)V_\mu V_\nu - pg_{\mu\nu}]$, where ρ , p and V^μ denotes, respectively, the rest mass density, pressure and velocity field of the fluid. Since in this approximation ρ and $\phi = \epsilon\varphi$ are small quantities, we can write $e^{-2\phi} \simeq 1 - 2\epsilon\varphi$, and thus, to first order in ϵ , we have $T_{\mu\nu} \simeq [(\rho c^2 + p)V_\mu V_\nu - pg_{\mu\nu}]$. On the other hand, in a non-relativistic regime we can neglect p with respect to ρ , which then implies $T \simeq \rho c^2$. Since in this approximation ρ and $\phi = \epsilon\varphi$ are small quantities, we have $T e^{(1-n/2)\phi} \simeq \rho c^2 (1 - \frac{\epsilon}{2}n\varphi) \simeq \rho c^2$. Thus we have

$$\frac{\epsilon}{2}\nabla^2 h_{00} = \left(\frac{n-3}{n-2} \right) k_n \rho c^2 \quad (26)$$

In the same approximation (9) becomes

$$\epsilon \nabla^2 \varphi = \frac{-4\kappa_n \rho c^2}{(n-1)(n-2) + 4\omega} . \quad (27)$$

From (24), (26) and (27) we finally get the n -dimensional Poisson's equation

$$\nabla^2 U = -K_n \rho \quad (28)$$

where $K_n = \kappa_n c^4 \left(\frac{n-3}{2(n-2)} + \frac{2}{(n-1)(n-2)+4\omega} \right)$ plays the role of the gravitational constant in n dimensions. At this point we recall that in n -dimensional general relativity the equation that corresponds to (28) is (see, for instance, [2])

$$\nabla^2 U = \left(\frac{n-3}{n-2} \right) k_n \rho c^4 . \quad (29)$$

For $n = 3$ the right-hand side of the above equation vanishes, hence the linearized Einstein theory fails to produce Newtonian gravity. However, due to the presence of the scalar field, $K_n \neq 0$. Therefore, WIST has a Newtonian limit for any $n \geq 3$.

VI. GEODESIC DEVIATION

An aspect of the strange behaviour of three-dimensional general relativity, first pointed out by Giddings et al ([2]), is the prediction that world lines of dust particles do not deviate. This is equivalent to saying that even if the space-time is allowed to have curvature these particles do not feel the gravitational interaction. Let us now investigate the same phenomenon in the light of Weyl integrable space-time theory.

Suppose that as the source of the gravitational field we have a pressureless perfect fluid ("dust"). In this case, the energy-momentum tensor of the fluid is given by

$$T^{\alpha\beta} = \rho u^\alpha u^\beta, \quad (30)$$

where $u^\alpha = u^\alpha(x)$ denotes the components of the 4-velocity field of the fluid particles. Let η^α denote the deviation vector of the congruence of geodesics determined by u^α . The equation of geodesic deviation is given by

$$\frac{D^2 \eta^\alpha}{ds^2} = \mathcal{R}^\alpha_{\mu\nu\kappa} u^\mu u^\kappa \eta^\nu, \quad (31)$$

where the operator $\frac{D}{ds}$ stands for the absolute derivative along the geodesic congruence. Now, from (25) the identity (15), the above equation may be written as

$$\begin{aligned} \frac{D^2 \eta^\alpha}{ds^2} = & \left(\frac{1+2\omega}{2} \right) (\phi_{,\mu} \phi_{,\nu} u^\mu u^\alpha \eta^\nu - \phi_{,\mu} \phi_{,\kappa} u^\mu u^\kappa \eta^\alpha - e^\phi \phi^{,\alpha} \phi_{,\nu} \eta^\nu + \phi^\alpha \phi_{,\kappa} u_\nu u^\kappa \eta^\nu - \frac{1}{2} \phi_{,\mu} \phi^{,\mu} u^\alpha u_\nu \eta^\nu \\ & + \frac{1}{2} e^\phi \phi_{,\mu} \phi^{,\mu} \eta^\alpha) - e^{-\phi} g_{\mu\lambda} \delta_{[\nu}^{[\alpha} Q_{\kappa]}^{\lambda]} u^\mu u^\kappa \eta^\nu. \end{aligned} \quad (32)$$

We thus see that in three-dimensional Weyl gravity the congruence of world lines of particles of a pressureless fluid has a non-vanishing geodesic deviation, so that a pair of freely falling particles will exhibit a relative accelerated motion, revealing the presence of a gravitational field. It is interesting to note that in this case the gravitational field manifests itself only through the Weyl scalar field ϕ .

VII. A STATIC AND CIRCULARLY SYMMETRIC SOLUTION

In this section we consider the problem of determining the gravitational field generated by a circularly symmetric matter distribution in a region outside the source. By solving the field equations we obtain a vacuum static solution, which, as we shall see, unlike three-dimensional general relativity, is not flat in regions where matter is absent.

Let us start by writing the line element of a circularly symmetric space-time in its most general form, which may be given by

$$ds^2 = e^{2N} dt^2 - e^{2P} dr^2 - r^2 d\theta^2, \quad (33)$$

where $N(r)$ and $P(r)$ are functions of the radial coordinate only. It is also assumed that the Weyl scalar field $\phi = \phi(r)$ also depends only on r . Then, the field equations (14) and (12) becomes

$$N'' + N'^2 - N'P' + \frac{N'}{r} = 0, \quad (34)$$

$$N'' + N'^2 - N'P' - \frac{P'}{r} = \lambda \phi'^2, \quad (35)$$

$$N' - P' = 0, \quad (36)$$

$$\phi'' + \phi'(N' - P') + \frac{\phi'}{r} = 0, \quad (37)$$

where prime denotes derivative with respect to r and $\lambda = -\frac{1}{2}(1+2\omega)$.

By substituting (36) into (37) we get

$$\phi'' + \frac{\phi'}{r} = 0,$$

whose general solution is given by

$$\phi = \phi_0 + A \ln r,$$

where ϕ_0 and A are integration constants. (Since in Weyl geometry the presence of an additive constant in the expression of the scalar field has no geometrical meaning it is convenient to set $\phi_0 = 0$.) On the other hand, also from (36), the equations (34) and (35) reduce, respectively, to

$$N'' + \frac{N'}{r} = 0$$

and

$$N'' - \frac{N'}{r} = \lambda \phi'^2.$$

It is easily seen that the above equations yield

$$N = N_0 + B \ln r ,$$

with $B = -\frac{\lambda}{2}A^2$, while (36) gives

$$P = P_0 + B \ln r ,$$

where N_0 and P_0 are integration constants.

By rescaling the time coordinate t we can set $N_0 = 0$. On the other hand, if we assume that there is no conical singularity in the space-time we can also take $P_0 = 0$. Finally, the line element (33) may be written as

$$ds^2 = r^{2B} dt^2 - r^{2B} dr^2 - r^2 d\theta^2 , \quad (38)$$

while the scalar field is given by

$$\phi = A \ln r . \quad (39)$$

Let us now apply the weak field limit to find the constant B in terms of the mass M of the matter distribution. To do this, we first note that when $B = 0$ (38) reduces to the metric of Minkowski space-time. From (39) it seems natural to identify the parameter ϵ of Section V with the constant A . For small values of B we can write $r^{2B} = e^{\ln r^{2B}} \simeq 1 + 2B \ln r$. At this point, let us recall that in two spatial dimensions the Newtonian gravitational potential $U(r)$ of a circularly symmetric mass distribution is given by $U(r) = G_2 M \ln r$, where G_2 denotes the gravitational constant in this dimensionality. Now, from (24) and recalling that $B = -\frac{\lambda}{2}A^2$ is a second-order term in $\epsilon = A$ we obtain $A = -\frac{2MG_2}{c^2}$ and $B = -\frac{2\lambda M^2 G_2^2}{c^4} = \frac{(1+2w)M^2 G_2^2}{c^4}$.

With respect to the space-time corresponding to this solution a few comments are in order. First, let us note that the basic invariants in this dimensionality are $I_1 = e^\phi \mathcal{R}$ and $I_2 = e^{2\phi} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}$. For the space-time (38) we have $I_1 = \frac{4B-A^2}{2} r^{A-2B-2}$ and $I_2 = F(A, B) r^{2(A-2B-2)}$, where $F(A, B)$ is a function of the constants A and B . Thus, it is not difficult to verify that this solution corresponds to a space-time presenting a naked singularity at the center of the matter distribution for values of $\omega > -\frac{1}{2} + \omega_0$, with $\omega_0 = \frac{A-2}{A^2}$. In fact, this result is not surprising since naked singularities are a common feature observed in (3+1)-dimensional general relativity models with a massless scalar fields [13].

VIII. FINAL REMARKS

Investigation in lower dimensional gravity has arisen essentially after the realization that the two or three-dimensional versions of Einstein gravity are rather peculiar and, in some sense, unsuitable for a theory of the gravitational interaction (for instance, gravitational waves do not exist in three-dimensional general relativity). However, the current motivation for this kind of research is that lower dimensional models can provide useful insights and ideas to construct a successful quantum theory of gravity [3]. Apart from general relativity, other theories of gravity have been studied in this context. Of much interest is the study of how black holes may form in gravity theories formulated in lower dimensions. Because general relativistic space-times are locally flat outside matter sources there are no black holes solutions, unless a negative cosmological constant is introduced, as was pointed out by Bañados *et al* some years ago [?]. On the other hand, different models of dilaton gravity also lead to the discovery of black hole solutions. In this work, we have approached the subject from the standpoint of Weyl integrable geometry, in which a scalar field plays the role of a geometrical field. We have found out that this modification in the space-time geometry leads to new features that are not present in three-dimensional Einstein gravity, such as the existence of a Newtonian limit and the non-vanishing geodesic deviation of the trajectories of freely falling particles. Finally, we have shown that another effect of the geometrical scalar field is that the space-time generated by a static circularly symmetric matter distribution corresponds, for certain values of the parameter ω , to a curved space-time corresponding with a naked singularity at the center of the distribution.

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- [1] See, for instance, J. D. Brown, *Lower Dimensional Gravity* (World Scientific 1988). See also S. Deser, R. Jackiw and G. 'T Hooft, *Ann. Phys.* **152**, 220 (1984). S. Deser and R. Jackiw, *Ann. Phys.* **153**, 405. A Staruszkiewicz, *Acta Phys. Polon.* **24**, 509 (1963). J. Richard Gott and Mark Alpert, *Gen. Rel. Grav.* **16**, 243 (1984). . M. Rocek and R. M. Williams, *Class. Quantum Grav.* **2**, 701 (1985). J. Richard Gott, J. Z. Simon and M. Alpert, *Gen. Rel. Grav.* **18**, 1019 (1986). V. B. Bezerra, *Class. Quantum Grav.* **5**, 1065 (1988). G. 'T Hooft, *Commun. Math. Phys.* **117**, 685 (1988). N. J. Cornish and N. E. Frankel, *Phys. Rev. D* **43**, 2555 (1991). G. 'T Hooft, *Class. Quantum Grav.* **9**, 1335 (1992). G. 'T Hooft, *Class. Quantum Grav.* **10**, 1023 (1993). J. P. S. Lemos, *Phys. Lett.* **B352**, 46 (1995). R. Jackiw, *Nucl. Phys.* **B252**, 343 (1985). S. Engineer, K. Srinivasan and T. Padmanabhan, *Astrophys. J.* **512**, 8 (1999).. R.B. Mann & S.F. Ross, *Phys. Rev. D* **47**, 3319 (1993).
 - [2] S. Giddings, J. Abbott and K. Kuchar, *Gen. Rel. Grav.* **16**, 751 (1984).
 - [3] For an overview see S. Carlip, *Quantum Gravity in 2+1 Dimensions* (Cambridge University Press, 1998).
 - [4] J. D. Barrow, A. B. Burd and D. Lancaster, *Class. Quantum Grav.* **3**, 551 (1986). J. L. Alonso, J. L. Cortés, and V. Laliena, *Phys. Rev.* **D67**, 024023 (2003). T. Kawai, *Phys. Rev.* **D48**, 5668 (1993). C. Romero and F. Dahia, *Int. J. Theor. Phys.* **33**, 2091 (1994).
 - [5] Y. Verbin, *Phys. Rev.* **D50**, 7318 (1994). S. Rippl, C. Romero and R. Tavakol, *Class. Quant. Grav.* **12**, 2411 (1995).
 - [6] M. Novello, L.A.R. Oliveira, J.M. Salim, E. Elbas, *Int. J. Mod. Phys.* **D1** (1993) 641-677. J. M. Salim and S. L. Sautú, *Class. Quant. Grav.* **13**, 353 (1996). H. P. de Oliveira, J. M. Salim and S. L. Sautú, *Class. Quant. Grav.* **14**, 2833 (1997). V. Melnikov, *Classical Solutions in Multidimensional Cosmology* in Proceedings of the VIII Brazilian School of Cosmology and Gravitation II (1995), edited by M. Novello (Editions Frontières) pp. 542-560, ISBN 2-86332-192-7. K.A. Bronnikov, M.Yu. Konstantinov, V.N. Melnikov, *Grav.Cosmol.* **1**, 60 (1995). J. Miritzis, *Class. Quantum .Grav.* **21**, 3043 (2004). J. Miritzis, *J.Phys. Conf. Ser.* **8**, 131 (2005).
 - [7] H. Weyl, *Sitzungesber Deutsch. Akad. Wiss. Berlin*, 465 (1918). H. Weyl, *Space, Time, Matter* (Dover, New York, 1952).
 - [8] For a historical account see A. Pais, *Subtle is the Lord*, Ch. 17 (Oxford University Press, Oxford, 1982). See, also, W. Pauli, *Theory of Relativity* (Dover, New York, 1981). P. G. Bergmann, *Theory of Relativity* (Dover, New York, 1976). L. O'Riافةartaigh and N. Straumann, *Rev. Mod. Phys.* **72**, 1 (2000).
 - [9] R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, Ch. 15, (McGraw-Hill, 1975).
 - [10] C. Romero, J. B. Fonseca-Neto, M. L. Pucheu, *Class. Quantum Grav.* **29**, 155015 (2012). T. S. Almeida, M. L. Pucheu, C. Romero and J. B. Formiga, *Phys. Rev. D* **89**, 064047.
 - [11] J. M. Overduin, B. Mashhoon and P. S. Wesson, *Astron. Astrophys.* **473**, 727 (2007).
 - [12] See, for instance, S. Weinberg, *Gravitation and Cosmology*, Ch. (Wiley, 1972).
 - [13] M. D. Roberts, *Gen. Rel. Grav.* **21**, 907 (1988).